($p = 3, 4, 6$) are:

$$N_0^{p2} = 309\, G_3^{3'} + 950\, G_3^{4'} + 953\, G_3^{6'} = 2212;$$

$$N_1^{p2} = 1634\, G_3^{1,3'} + 6361\, G_3^{1,4'} + 7288\, G_3^{1,6'} = 15283;$$

$$N_2^{p2} = 13391\, G_3^{2,3'} + 53664\, G_3^{2,4'} + 78825\, G_3^{2,6'} = 145880;$$

$$N_3^{p2} = 150197\, G_3^{3,3'} + 441924\, G_3^{3,4'} + 967568\, G_3^{3,6'} = 1559689;$$

$$N_4^{p2} = 1888320\, G_3^{4,3'} + 2056320\, G_3^{4,4'} + 10321920\, G_3^{4,6'} = 14266560;$$

$$N_5^{p2} = 19998720\, G_3^{5,3'} = 19998720.$$

The possible physical applications of the generalized colored symmetry groups derived are considered by Koptsik (1988).

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Mackay Groups

BY S. V. JABLAN

The Mathematical Institute, Knez Mihailova 35, *Belgrade, Yugoslavia*

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Abstract

The number of junior Mackay groups of M^m type is calculated for different nonisomorphic antisymmetric characteristics formed by $1 \leq l \leq 4$ generators. Combinatorial relationships connecting Mackay and Zamorzaev multiple-antisymmetry groups are established.

The idea, originated by Speiser (1927) and realized by Weber (1929), of representing symmetry groups of bands by black-and-white plane diagrams was the starting point for introducing antisymmetry (Heesch, 1929). The color change white-black used as the possibility for the dimensional transition from the symmetry groups of friezes G_{21} to the symmetry groups of bands G_{321} or from the plane groups G_2 to the layer groups G_{32}, applied on Fedorov space groups G_3 to derive the hyperlayer-symmetry groups G_{43} (Heesch, 1930), was the beginning of the theory of antisymmetry. Its simple mathematical explanation is the following: if G is a discrete symmetry group with the anti-identity transformation e_1 satisfying the relationship $e_1{}^2 = E$ and commuting with every symmetry S from G, the group G^1, consisting of transformations S^1 ($S^1 = S$ or $S^1 = e_1 S$), is an antisymmetry group. The antisymmetry group G^1 can be the generating ($G_1 = G$), the senior ($G^1 = G \times C_2 = G \times \{e_1\}$) or the junior ($G^1 \simeq G$) group. Every junior antisymmetry group G^1 is uniquely defined by the generating symmetry group G and its subgroup H of index 2, the symmetry subgroup of G^1, *i.e.* by the symbol G/H ($G/H \simeq C_2 = \{e_1\}$). The anti-identity transformation e_1 can be interpreted as the change of any physical or geometrical bivalent property [*e.g.* (+ −), ($S\,N$), (convex concave) *etc.*] independent of the symmetry group G. The development of the theory of antisymmetry can be followed through the works of Shubnikov *et al.* (1964), Shubnikov & Koptsik (1974) and Zamorzaev (1976).

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Its natural generalization, multiple antisymmetry, was suggested by Shubnikov (1945) and introduced by Zamorzaev & Sokolov (1957). Three months later, a different concept of multiple antisymmetry was proposed by Mackay (1957). During the next 30 years, mostly due to the contribution of the Kishinev school (Zamorzaev, Palistrant, Galyarskii, ...), the theory of multiple antisymmetry became an integral part of mathematical crystallography and acquired the status of a complete theory extended to all categories of isometric symmetry groups of the space E^n ($n \leq 3$), different kinds of nonisometric symmetry groups (of similarity symmetry, conformal symmetry *etc.*) and P-symmetry groups (Zamorzaev, 1976; Zamorzaev, Galyarskii & Palistrant, 1978; Zamorzaev & Palistrant, 1980; Zamorzaev, Karpova, Lungu & Palistrant, 1986; Zamorzaev, 1988). On the other hand, investigation of the Mackay approach to multiple antisymmetry (Mackay, 1957; Nowacki, 1960; Wondratschek & Niggli, 1961; Zamorzaev, 1976) was not continued.

Let G be a discrete symmetry group and e_i ($1 \leq i \leq l$) be the anti-identities satisfying the relationships $e_i^2 = E$, $e_i e_j = e_j e_i$ and commuting with all elements of G. The group consisting of transformations $S' = e'S$, where e' is the identity, anti-identity or some product of anti-identities, is called the multiple-antisymmetry group. In this paper we will consider only the junior multiple-antisymmetry groups of the M^m type, *i.e.* the multiple-antisymmetry groups isomorphic with their generating symmetry group and possessing an independent system of antisymmetries (Jablan, 1986). Every junior multiple-antisymmetry group G' of the M^m type can be uniquely defined by the extended group/subgroup symbol $G/(H_1, \ldots, H_m)/H$, where G is the generating group, H_i are its subgroups of index 2 satisfying the relationships $G/H_i \simeq C_2 = \{e_i\}$ ($1 \leq i \leq m$) and H is the subgroup of G of index 2^m, the symmetry subgroup of G' ($G/H \simeq C_2{}^m = \{e_1\} \times \ldots \times \{e_m\}$).

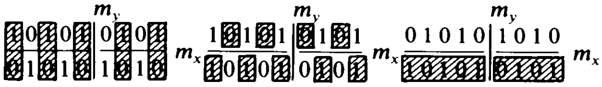
To establish the equality of multiple-antisymmetry groups, three different criteria can be used.

(1) The 'strong' equality criterion, according to which the anti-identities e_i are nonequivalent. Consequently, in the symbol $G/(H_1, \ldots, H_m)/H$, the order of the subgroups $H_1, \ldots, H_m$ is important. This means that the bivalent changes e_m are physically different (nonequivalent) [*e.g.* (white black), (+ −), (S N), (0 1) ...].

(2) The 'medium' equality criterion, where all e_m are treated as equivalent (*i.e.* permutable), so the order of the subgroups mentioned is not important.

(3) The 'weak' equality criterion, G/H.

By use of the 'strong' equality criterion, we have as a result Zamorzaev groups (Z groups); by use of the 'medium' equality criterion, we obtain Mackay (or compound) multiple-antisymmetry groups (M groups). An illustration of the criteria mentioned is given by the 2-multiple antisymmetry groups generated from the symmetry group of friezes $pmm2$, where $e_1 = (0\ 1)$ and $e_2 = (▨)$.



(*a*) (*b*) (*c*)

If the bivalent changes e_1 and e_2 are treated as nonequivalent, we have 'strong' equality and three different Z groups: (*a*) $pmm2/(p112, pm11)/p111$, (*b*) $pmm2/(pm11, p112)/p111$, (*c*) $pmm2/(p112, p1m1)/p111$. If the bivalent changes are treated as equivalent, we have 'medium' equality and two M groups: $(a) = (b)$ $pmm2/(p112, pm11)/p111 = pmm2/(pm11, p112)/p111$, (*c*) $pmm2/(p112, p1m1)/p111$. According to the 'weak' equality criterion, $(a) = (b) = (c)$.

With the positional symbols (Koptsik, 1975), the same 2-multiple antisymmetry groups can be denoted as (*a*) $p^{(1,1)}m^{(1',1)}m^{(1',1')}2^{(1,1')}$, (*b*) $p^{(1,1)}m^{(1,1')}m^{(1',1')}2^{(1',1)}$, (*c*) $p^{(1,1)}m^{(1',1')}m^{(1',1)}2^{(1,1')}$. Permutation of anti-identities e_1, e_2 and the induced permutation of the subgroups $H_1 = p112$, $H_2 = pm11$ results in the transformation $(1,1) \leftrightarrow (1,1)$, $(1',1) \leftrightarrow (1,1')$, $(1',1') \leftrightarrow (1',1')$, transforming the positional symbol of group (*a*) into the symbol of group (*b*), showing that groups (*a*) and (*b*) are different as Z groups and equal as M groups.

A very efficient method for the derivation of Z groups is the antisymmetric-characteristic method (Jablan, 1986, 1987, 1990).

Definition 1: Let all products of generators of a group G, within which every generator participates once at the most, be formed and then subsets of transformations that are equivalent in the sense of symmetry with regard to the symmetry group G be separated. The resulting system is called the antisymmetric characteristic of the group G [AC(G)].

Theorem 1: Two Z groups G' and G'' of M^m type for fixed m, with common generating group G, are equal if and only if they possess equal AC.

Every AC(G) completely defines the series $N_m(G)$, where $N_m(G)$ denotes the number of Z groups of M^m type derived from G for m fixed ($1 \leq m \leq l$).

Theorem 2: Symmetry groups possessing isomorphic AC generate the same number of Z or M groups of M^m type for each particular m ($1 \leq m \leq l$); these groups correspond with each other with regard to structure.

Corollary: the derivation of all Z or M groups can be completely reduced to the construction of all nonisomorphic AC and derivation of the corresponding groups of M^m type from these AC.

In the sense of Z groups, nonisomorphic antisymmetry characteristics formed by $1 \leq l \leq 4$ generators were investigated by Jablan (1990). They are listed below.

$l=1$
1.1 $\{A\}$.

$l=2$
2.1 $\{A\}\{B\}$;
2.2 $\{A, B\}$;
2.3 $\{A, B, AB\}$.

$l=3$
3.1 $\{A\}\{B\}\{C\}$;
3.2 $\{A, B\}\{C\}$;
3.3 $(A, B, C, AB, AC, BC, ABC)$;
3.4 $\{A, B\}\{C, ABC\}$;
3.5 (A, B, C);
3.6 (A, B, C, ABC);
3.7 $\{A, B, C\}$;
3.8 $\{\{A, B\}, \{C, ABC\}\}$;
3.9 $\{A, B, C, ABC\}$;
3.10 $\{A, B, C, AB, AC, BC, ABC\}$.

$l=4$
4.1 $\{A\}\{B\}\{C\}\{D\}$;
4.2 $\{A, B\}\{C\}\{D\}$;
4.3 $([A, B], [C, ABC], [D, ABD], [AC, BC], [AD, BD]\ [CD, ABCD], [ACD, BCD])$;
4.4 $\{A, B\}\{C, D\}\{AC, BD\}$;
4.5 $\{A\}\{B, C\}\{D, BCD\}$;
4.6 $\{A, B\}\{C, D\}$;
4.7 $\{B, AB\}\{C, AC\}\{D, AD\}$;
4.8 $\{A\}(B, C, D)$;
4.9 $(/A, B/, /C, ABC/, /D, ABD/, /ACD, BCD/)$;
4.10 $\{A, B, C\}\{D\}$;
4.11 $\{\{A, B\}, \{CA, CB\}\}\{D, CD\}$;
4.12 $\{[A, B], [C, D]\}$;
4.13 $\{\{B, AB\}, \{C, AC\}\}\{D, AD\}$;
4.14 (A, B, C, D);
4.15 $(C, A, CA)\{(B, C, ABC), (BD, BCD, ABCD)\}$;
4.16 $\{\{A, B\}, \{C, D\}\}$;
4.17 $(\{A, B\}, \{C, ABC\}, \{D, ABD\}, \{AC, BC\}, \{AD, BD\}, \{CD, ABCD\}, \{ACD, BCD\})$;
4.18 $\{A, B, AB\}\{C, D\}$;
4.19 $\{A, B, C, ABC\}\{D\}$;
4.20 $\{\{A, B\}, \{C, ABC\}\}\{\{D, ABD\}, \{ACD, BCD\}\}$;
4.21 $(\{A, AD\}, \{B, BD\}, \{C, CD\})$;
4.22 $\{A, B, C, D\}$;
4.23 $(\{A, B\}, \{C, ABC\}\}\{\{D, ABD\}, \{ACD, BCD\})$;
4.24 $\{\{B, AB\}, \{C, AC\}, \{D, AD\}\}$;
4.25 $\{\{\{A, B\}, \{C, ABC\}\}, \{\{D, ABD\}, \{ACD, BCD\}\}\}$;
4.26 $\{A, B, C, ABC\}\{D, ABD, ACD, BCD\}$;
4.27 $\{\{A, B\}, \{C, D\}, \{AC, BD\}\}$;
4.28 $\{\{A, B\}, \{C, ABC\}, \{D, ABD\}, \{ACD, BCD\}\}$;
4.29 $\{A, B, C, D, ABC, ABD, ACD, BCD\}$;
4.30 $\{A, B, C, D, AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, ABCD\}$.

there, round braces ( ) denote cyclic permutation of the enclosed elements, square braces [ ] denote simultaneous commutation of elements; the elements between // remain fixed in their places.

Definition 2: Two or more Z or M groups belong to a family if and only if they are derived from the same symmetry group G.

Theorem 3: Two M groups G' and G'' of M^m type belonging to the same family are equal if and only if there is a permutation of the anti-identities $e_1, \dots, e_m$ transforming AC(G') into AC(G'').

In addition, every AC(G) completely defines the series $M_m(G)$, where $M_m(G)$ denotes the number of M groups of M^m type derived from G, for m fixed $(1 \le m \le l)$. Of course, $M_1(G) = N_1(G)$.

To find the series M_m corresponding to all aforementioned nonisomorphic antisymmetric characteristics for $1 \le l \le 4$, we first need to find all nonisomorphic systems of independent anti-identities e_m and their products for $2 \le m \le l$, *i.e.* the systems from which every anti-identity can be obtained as independent by multiplying the suitably chosen elements of the system. For $m=2$, $l=2$ there are two such systems:

(1) e_1, e_2;
(2) e_1, e_1e_2.

For $l=3$ and $m=2$ there are six such systems:

(1) E, e_1, e_1e_2;
(2) E, e_1, e_2;
(3) e_1, e_1, e_2;
(4) e_1, e_1, e_1e_2;
(5) e_1, e_2, e_1e_2;
(6) e_1, e_1e_2, e_1e_2.

For $l=3$ and $m=3$ there are seven such systems:

(1) e_1, e_2, e_3;
(2) e_1e_3, e_2, e_3;
(3) e_1e_3, e_2e_3, e_3;
(4) e_1e_2, e_2e_3, e_3;
(5) $e_1e_2e_3, e_2, e_3$;
(6) $e_1e_2e_3, e_2e_3, e_3$;
(7) $e_1e_2e_3, e_1e_3, e_2e_3$.

For $l=4$ and $m=2$ there are 13 such systems:

(1) E, E, e_1, e_2;
(2) E, e_1, e_1, e_2;
(3) e_1, e_1, e_2, e_2;
(4) e_1, e_1, e_1, e_2;
(5) E, E, e_1, e_1e_2;
(6) E, e_1, e_1, e_1e_2;
(7) e_1, e_1, e_1e_2, e_1e_2;
(8) e_1, e_1, e_1, e_1e_2;
(9) E, e_1, e_2, e_1e_2;
(10) e_1, e_2, e_1e_2, e_1e_2;
(11) e_1, e_1, e_2, e_1e_2;
(12) E, e_1, e_1e_2, e_1e_2;
(13) $e_1, e_1e_2, e_1e_2, e_1e_2$.

For $l=4$, $m=3$ there are 33 such systems:

(1) E, e_1, e_2, e_1e_3;
(2) E, e_1e_2, e_2e_3;
(3) $E, e_1, e_1e_2, e_1e_2e_3$;
(4) e_1, e_2, e_1e_2, e_1e_3;
(5) $e_1, e_2, e_1e_3, e_1e_2e_3$;
(6) $e_1, e_1e_2, e_2e_3, e_1e_2e_3$;
(7) $E, e_1, e_2, e_1e_2e_3$;
(8) E, e_1, e_1e_2, e_1e_3;
(9) $E, e_1e_2, e_1e_3, e_1e_2e_3$;
(10) e_1, e_2, e_3, e_1e_2;
(11) e_1, e_2, e_2, e_1e_3;
(12) e_1, e_2, e_2, e_2e_3;
(13) $e_1, e_2, e_2, e_1e_2e_3$;
(14) e_1, e_2, e_1e_3, e_2e_3;
(15) e_1, e_2, e_1e_3, e_1e_3;
(16) $e_1, e_2, e_1e_2, e_1e_2e_3$;
(17) e_1, e_1, e_1e_2, e_2e_3;
(18) $e_1, e_1, e_1e_2, e_1e_2e_3$;
(19) $e_1e_2, e_1e_3, e_1e_3, e_1e_2e_3$;
(20) $e_1, e_1e_2, e_1e_3, e_2e_3$;
(21) $e_1, e_1e_2, e_1e_2, e_1e_3$;
(22) $e_1, e_1e_2, e_1e_2, e_2e_3$;
(23) $e_1, e_1e_2, e_1e_2, e_1e_2e_3$;
(24) $e_1, e_1e_2, e_1e_3, e_1e_2e_3$;
(25) $e_1, e_1e_2, e_1e_2e_3, e_1e_2e_3$;
(26) $e_1, e_1e_2, e_2e_3, e_2e_3$;
(27) e_1, e_2, e_3, e_3;
(28) $e_1, e_2, e_1e_2e_3, e_1e_2e_3$;
(29) e_1, e_1, e_1e_2, e_1e_3;
(30) $e_1e_2, e_1e_3, e_1e_2e_3, e_1e_2e_3$;
(31) E, e_1, e_2, e_3;
(32) $e_1, e_2, e_3, e_1e_2e_3$;
(33) $e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3$.

For $l=4$, $m=4$ there are 51 such systems:

(1) e_1, e_2, e_3, e_4;
(2) e_1, e_2, e_3, e_1e_4;
(3) $e_1, e_2, e_3, e_1e_2e_4$;
(4) $e_1, e_2, e_3, e_1e_2e_3e_4$;
(5) e_1, e_2, e_1e_3, e_1e_4;
(6) e_1, e_2, e_1e_3, e_2e_4;
(7) e_1, e_2, e_1e_3, e_3e_4;
(8) $e_1, e_2, e_1e_3, e_1e_2e_4$;
(9) $e_1, e_2, e_1e_3, e_1e_3e_4$;
(10) $e_1, e_2, e_1e_3, e_2e_3e_4$;
(11) $e_1, e_2, e_1e_3, e_1e_2e_3e_4$;
(12) $e_1, e_2, e_3e_4, e_1e_2e_3$;
(13) $e_1, e_2, e_1e_2e_3, e_1e_2e_4$;
(14) $e_1, e_2, e_1e_2e_3, e_1e_3e_4$;
(15) $e_1, e_2, e_1e_2e_3, e_1e_2e_3e_4$;
(16) $e_1, e_1e_2, e_1e_3, e_1e_4$;
(17) $e_1, e_1e_2, e_1e_3, e_2e_4$;
(18) $e_1, e_1e_2, e_1e_3, e_1e_2e_4$;
(19) $e_1, e_1e_2, e_1e_3, e_2e_3e_4$;
(20) $e_1, e_1e_2, e_1e_3, e_1e_2e_3e_4$;
(21) $e_1, e_1e_2, e_2e_3, e_2e_4$;
(22) $e_1, e_1e_2, e_2e_3, e_3e_4$;
(23) $e_1, e_1e_2, e_2e_3, e_1e_2e_4$;
(24) $e_1, e_1e_2, e_2e_3, e_1e_3e_4$;
(25) $e_1, e_1e_2, e_2e_3, e_2e_3e_4$;
(26) $e_1, e_1e_2, e_2e_3, e_1e_2e_3e_4$;
(27) $e_1, e_1e_2, e_3e_4, e_1e_2e_3$;
(28) $e_1, e_1e_2, e_1e_2e_3, e_1e_2e_4$;
(29) $e_1, e_1e_2, e_1e_2e_3, e_1e_3e_4$;
(30) $e_1, e_1e_2, e_1e_2e_3, e_2e_3e_4$;
(31) $e_1, e_1e_2, e_1e_2e_3, e_1e_2e_3e_4$;
(32) $e_1, e_2e_3, e_1e_2e_4, e_2e_3e_4$;
(33) $e_1, e_2e_3, e_1e_2e_4, e_1e_2e_3e_4$;
(34) $e_1, e_1e_2e_3, e_1e_2e_4, e_2e_3e_4$;
(35) $e_1, e_1e_2e_3, e_1e_2e_4, e_1e_2e_3e_4$;
(36) $e_1e_2, e_1e_3, e_1e_4, e_1e_2e_3$;
(37) $e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4$;
(38) $e_1e_2, e_1e_3, e_2e_4, e_1e_2e_3$;
(39) $e_1e_2, e_1e_3, e_2e_4, e_1e_3e_4$;
(40) $e_1e_2, e_1e_3, e_1e_2e_3, e_1e_2e_4$;
(41) $e_1e_2, e_1e_3, e_1e_2e_3, e_2e_3e_4$;
(42) $e_1e_2, e_1e_3, e_1e_2e_3, e_1e_2e_3e_4$;
(43) $e_1e_2, e_1e_3, e_1e_2e_4, e_1e_2e_3e_4$;
(44) $e_1e_2, e_1e_3, e_2e_3e_4, e_1e_2e_3e_4$;
(45) $e_1e_2, e_3e_4, e_1e_2e_3, e_1e_3e_4$;
(46) $e_1, e_2e_3, e_2e_4, e_2e_3e_4$;
(47) $e_1e_2, e_1e_2e_3, e_1e_2e_4, e_1e_3e_4$;
(48) $e_1e_2, e_1e_2e_3, e_1e_3e_4, e_1e_2e_3e_4$;
(49) $e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_1e_2e_3e_4$;
(50) $e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4$;
(51) $e_1, e_2e_3, e_2e_4, e_1e_2e_3e_4$.

A further procedure is illustrated by derivation of the M groups for $m=2$, $l=2$. The generators A, B in the maximal AC 2.1 are replaced by the elements from the corresponding system:

2.1 $\{A\}\{B\}$
$\{e_1\}\{e_2\}$ (1);
$\{e_2\}\{e_1\}$ (1);
$\{e_1\}\{e_1e_2\}$ (2);
$\{e_1e_2\}\{e_1\}$ (3).

From this, we have three M groups derived from the maximal AC 2.1. From each of them, by permutation of the anti-identities, we obtain two Z groups. Hence, $N_2(2.1)=2M_2(2.1)$. The generators A, B in other anti-symmetric characteristics are then replaced by the anti-identities and their products from the three M groups corresponding to AC 2.1. From AC 2.2 we obtain two M groups:

2.2 $\{A\}\{B\}$
$\{e_1, e_2\}$ (1);
$\{e_1, e_1e_2\}$ (2).

By permutation of the anti-identities e_1 and e_2, the first AC is transformed into itself, giving only one Z group, and the other gives two Z groups. Hence, $M_2(2.2)=2$, $N_2(2.2)=1\times 2+1\times 1=3$. With the same substitution, we obtain from the AC 2.3 only one M group:

2.3 $\{A, B, C\}$
$\{e_1, e_2, e_1e_2\}$ (1).

By permutation of the anti-identities e_1 and e_2, this AC is transformed into itself, giving only one Z group. Hence, $M_2(2.3) = N_2(2.3) = 1$. As a result, we obtain the table below of the numbers N_m and M_m for all antisymmetric characteristics formed by $l = 2$ generators:

	N_1	N_2	M_1	M_2
2.1	3	6	3	3
2.2	2	3	2	2
2.3	1	1	1	1

and the combinatorial relationships connecting the corresponding numbers N_2 and M_2, showing that all Z groups can be obtained from the corresponding M groups by permuting the anti-identities:

$N_2(2.1) = 3 \times 2 = 6$;
$N_2(2.2) = 1 \times 2 + 1 \times 1 = 3$;
$N_2(2.3) = 1 \times 1 = 1$.

All the combinatorial relations connecting the numbers M_m and N_m are also the double control for the numbers N_m obtained in the work by Jablan (1990).

The same procedure is realized for $l = 3$ and $l = 4$, $m \leq l$. As a result we obtain the table below of the numbers N_m and M_m for all antisymmetric characteristics formed by $l = 3$ generators:

	N_1	N_2	N_3	M_1	M_2	M_3
3.1	7	42	168	7	21	28
3.2	5	24	84	5	13	16
3.3	4	24	96	4	12	16
3.4	4	15	42	4	9	10
3.5	3	14	56	3	7	10
3.6	3	12	42	3	7	8
3.7	3	10	28	3	6	7
3.8	3	9	21	3	6	6
3.9	2	4	7	2	3	3
3.10	1	1	1	1	1	1

and the combinatorial relationships connecting the corresponding numbers N_m and M_m:

$N_2(3.1) = 21 \times 2 = 42$;
$N_3(3.1) = 28 \times 6 = 168$;
$N_2(3.2) = 11 \times 2 + 2 \times 1 = 24$;
$N_3(3.2) = 12 \times 6 + 4 \times 3 = 84$;
$N_2(3.3) = 12 \times 2 = 24$;
$N_3(3.3) = 16 \times 6 = 96$;
$N_2(3.4) = 6 \times 2 + 3 \times 1 = 15$;
$N_3(3.4) = 4 \times 6 + 6 \times 3 = 42$;
$N_2(3.5) = 7 \times 2 = 14$;
$N_3(3.5) = 9 \times 6 + 1 \times 2 = 56$;
$N_2(3.6) = 5 \times 2 + 2 \times 1 = 12$;
$N_3(3.6) = 6 \times 6 + 2 \times 3 = 42$;
$N_2(3.7) = 4 \times 2 + 2 \times 1 = 10$;
$N_3(3.7) = 3 \times 6 + 3 \times 3 + 1 \times 1 = 28$;
$N_2(3.8) = 3 \times 2 + 3 \times 1 = 9$;
$N_3(3.8) = 1 \times 6 + 5 \times 3 = 21$;
$N_2(3.9) = 1 \times 2 + 2 \times 1 = 4$;
$N_3(3.9) = 2 \times 3 + 1 \times 1 = 7$;
$N_2(3.10) = 1 \times 1 = 1$;
$N_3(3.10) = 1 \times 1 = 1$.

For $l = 4$ we have:

	N_1	N_2	N_3	N_4	M_1	M_2	M_3	M_4
4.1	15	210	2520	20160	15	105	420	840
4.2	11	126	1344	10080	11	65	236	444
4.3	9	120	1440	11520	9	60	240	480
4.4	9	108	1260	10080	9	57	216	426
4.5	9	84	756	5040	9	45	144	282
4.6	8	75	714	5040	8	41	134	237
4.7	8	63	462	2520	8	35	98	147
4.8	7	74	840	6720	7	37	142	284
4.9	7	66	672	5040	7	35	120	222
4.10	7	58	504	3360	7	31	97	164
4.11	7	54	420	2520	7	31	88	138
4.12	6	57	630	5040	6	33	114	213
4.13	6	39	252	1260	6	23	58	81
4.14	5	54	630	5040	5	29	108	214
4.15	5	44	448	3360	5	23	81	140
4.16	5	39	357	2520	5	23	70	122
4.17	5	36	264	1440	5	21	80	102
4.18	5	34	266	1680	5	20	56	90
4.19	5	28	168	840	5	16	39	55
4.20	5	27	147	630	5	17	38	47
4.21	4	23	154	840	4	13	34	51
4.22	4	22	147	840	4	13	33	51
4.23	4	21	126	630	4	13	28	41
4.24	4	19	98	420	4	12	26	33
4.25	4	18	84	315	4	12	24	28
4.26	4	16	63	210	4	10	19	22
4.27	3	21	210	1680	3	14	42	77
4.28	3	10	35	105	3	7	12	13
4.29	2	4	8	15	2	3	4	4
4.30	1	1	1	1	1	1	1	1

The combinatorial relationships connecting the numbers N_m and M_m for $l = 4$ can be established in the same way as before.

The use of the results obtained for calculating the numbers of Z groups N_m and M groups M_m for some well known categories of symmetry groups is illustrated by the symmetry groups of friezes G_{21}. There are seven symmetry groups of friezes, given by their crystallographic symbol, generators, antisymmetry characteristic and the number of equivalence classes according to the relation of AC isomorphism (Jablan, 1990):

$p111$	$\{\vec{b}\}$	$\{\vec{b}\}$	1.1
$pb11$	$\{b\}$	$\{b\}$	1.1
$pm11$	$\{\vec{b}, m_x\}$	$\{\vec{b}\}\{m_x\}$	2.1

*pbm*2	$\{m_y, 2_z\}$	$\{m_y\}\{2_z\}$	2.1
*p*1*m*1	$\{\vec{b}, m_y\}$	$\{m_y, \vec{b}m_y\}$	2.2
*p*112	$\{\vec{b}, 2z\}$	$\{2_z, 2_z\vec{b}\}$	2.2
*pmm*2	$\{\vec{b}, m_x, 2_z\}$	$\{m_x\}\{2_z m_x, 2_z m_x b\}$	3.2

where translations and glide reflections are denoted by $\vec{b}$ and b respectively, reflections parallel to the x and y-axes are denoted by m_x and m_y respectively and half-turns are denoted by 2_z.

For $l=1$ we have the 17 well known black-white symmetry groups of friezes. For $l=2$, the M groups derived are distributed into families and are given by their antisymmetric characteristics and extended group/subgroup symbols.

*pm*11	(1)	$\{e_1\}\{e_2\}$	*pm*11/(*pm*11, *p*111)/*p*111
	(2)	$\{e_1\}\{e_1e_2\}$	*pm*11/(*pb*11, *p*111)/*p*111
	(3)	$\{e_1e_2\}\{e_1\}$	*pm*11/(*pb*11, *pm*11)/*p*111
*pbm*2	(1)	$\{e_1\}\{e_2\}$	*pbm*2/(*p*1*m*1, *p*112)/*p*111
	(2)	$\{e_1\}\{e_1e_2\}$	*pbm*2/(*pb*11, *p*112)/*p*111
	(3)	$\{e_1e_2\}\{e_1\}$	*pbm*2/(*pb*11, *p*1*m*1)/*p*111
*p*1*m*1	(1)	$\{e_1, e_2\}$	*p*1*m*1/(*p*1*m*1, *p*1*m*1)/*p*111
	(2)	$\{e_1e_2, e_1\}$	*p*1*m*1/(*p*1*m*1, *p*111)/*p*111
*p*112	(1)	$\{e_1, e_2\}$	*p*112/(*p*112, *p*112)/*p*111
	(2)	$\{e_1e_2, e_1\}$	*p*112/(*p*112, *p*111)/*p*111
*pmm*2	(1)	$\{E\}\{e_1, e_2\}$	*pmm*2/(*pmm*2, *pmm*2)/*pm*11
	(2)	$\{e_1e_2\}\{e_1, e_2\}$	*pmm*2/(*pbm*2, *pbm*2)/*pb*11
	(3)	$\{e_1\}\{E, e_2\}$	*pmm*2/(*pmm*2, *p*1*m*1)/*p*1*m*1
	(4)	$\{E\}\{e_1, e_1e_2\}$	*pmm*2/(*pmm*2, *pm*11)/*pm*11
	(5)	$\{e_1\}\{E, e_1e_2\}$	*pmm*2/(*pmm*2, *pbm*2)/*p*1*m*1
	(6)	$\{e_1e_2\}\{E, e_1\}$	*pmm*2/(*pbm*2, *p*1*m*1)/*p*1*m*1
	(7)	$\{e_1\}\{e_1, e_2\}$	*pmm*2/(*pmm*2, *pbm*2)/*p*112
	(8)	$\{e_1\}\{e_2, e_2\}$	*pmm*2/(*p*1*m*1, *pm*11)/*p*111
	(9)	$\{e_1\}\{e_1, e_1e_2\}$	*pmm*2/(*pmm*2, *p*112)/*p*112
	(10)	$\{e_1e_2\}\{e_1, e_1\}$	*pmm*2/(*p*1*m*1, *p*112)/*p*111
	(11)	$\{e_1\}\{e_2, e_1e_2\}$	*pmm*2/(*pbm*2, *pm*11)/*pb*11
	(12)	$\{e_1\}\{e_1e_2, e_1e_2\}$	*pmm*2/(*p*112, *pm*11)/*p*111
	(13)	$\{e_1e_2\}\{e_1, e_1e_2\}$	*pmm*2/(*pbm*2, *p*112)/*p*112

By permutation of the anti-identities from the first group of the third and fourth family and from the first two groups of the fifth family, we obtain one Z group and from each of the other groups we obtain two Z groups. Hence, $M_2(G_{21})=23$, $N_2(G_{21})=4\times2+19\times1=27$.

In the same way, for $l=3$ we obtain from the generating symmetry group *pmm*2 16 M groups. By permutations of anti-identities from each of the four groups

(1)	$\{e_1\}\{e_2, e_3\}$	*pmm*2/(*pmm*2, *pmm*2, *p*1*m*1)/*p*111
(2)	$\{e_1\}\{e_1e_2, e_1e_3\}$	*pmm*2/(*pmm*2, *pmm*2, *p*112)/*p*111
(3)	$\{e_1e_2e_3\}\{e_1, e_2\}$	*pmm*2/(*pbm*2, *pbm*2, *p*1*m*1)/*p*111
(4)	$\{e_1e_2e_3\}\{e_1e_3, e_2e_3\}$	*pmm*2/(*pbm*2, *pbm*2, *p*112)/*p*111,

we obtain three Z groups and from each of the other groups we obtain six Z groups. Hence, $M_3(G_{21})=16$, $N_3(G_{21})=12\times6+4\times3=84$.

Using the table of results, we can easily calculate the numbers M_m for different categories of symmetry groups, knowing only the antisymmetric characteristics of the member groups. For example, in the case of plane symmetry groups G_2, antisymmetric characteristics of the groups *cm*, *p*4*g* and *p*6*m* belong to the equivalence class 2.1 according to the relation of AC isomorphism, AC of the groups *pg*, *pgg*, *p*4 belong to the class 2.2, AC of the group *p*1 belong to the class 2.3, AC of the groups *pm*, *pmg*, *cmm*, *p*4*m* belong to the class 3.2, AC of the group *p*2 belong to the class 3.9 and AC of the group *pmm* belong to the class 4.16. M groups cannot be derived from the remaining plane symmetry groups *p*3, *p*31*m*, *p*3*m*1, *p*6 for $l\geq2$. For $l=1$ we have the well known 46 black-white groups, $M_2(G_2)=94$, $M_3(G_2)=137$ and $M_4(G_2)=122$.

By permuting the anti-identities, we may obtain from them the corresponding Z groups of M^m type derived from plane-symmetry groups G_2 (Zamorzaev, 1976), where the following combinatorial relationships connecting numbers M_m and N_m hold:

$$N_2(G_2)=73\times2+21\times1=167;$$
$$N_3(G_2)=97\times6+39\times3+1\times1=700;$$
$$N_4(G_2)=90\times24+29\times12+1\times6+2\times3=2520.$$

Different physical applications of Z and M groups can be constructed according to Koptsik (1988).

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